IES / ISS EXAM, 2010

Sl. No. 1131

C-HLR-K-TA

STATISTICS – I

Time Allowed : Three Hours	*Maximum Marks : 200*

INSTRUCTIONS

Candidates should attempt FIVE questions in ALL including Question Nos. 1 and 5 which are compulsory. The remaining THREE questions should be answered by choosing at least ONE question each from Section A and Section B.

The number of marks carried by each question is indicated against each.

Answers must be written only in ENGLISH.

(Symbols and abbreviations are as usual)

Any essential data assumed by candidates for answering questions must be clearly stated.

Section – A

1. Answer any *five* of the following : 8×5=40

(a) An unbiased die is rolled twice. Let A be the event that the first throw shows a number ≤ 2, and B be the event that the second throw shows at least 5.

Show that $P(A \cup B) = \frac{5}{9}$.

(Contd.)

(b) Let $\{A_n\}$ be a non-decreasing sequence of events. Show that :

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right).$$

(c) Let X be a random variable defined on (Ω, A, P). Define a point function $F(x) = P\{\omega : X(\omega) \leq x\}$, for all $x \in R$. Show that the function F is indeed a distribution function.

(d) Let $k > 0$ be a constant, and

$$f(x) = \begin{cases} kx(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

obtain $P(X > 0 \cdot 3)$.

(e) Let X be a random variable with $E(X) = 0$ and $\text{Var}(X) = \sigma^2$.

Show that

$$P(X \geq x) \leq \frac{\sigma^2}{\sigma^2 + x^2} \text{ if } x > 0$$

and $$P(X \geq x) \geq \frac{x^2}{\sigma^2 + x^2} \text{ if } x < 0.$$

(f) Let $X_1, X_2, \ldots\ldots\ldots$, be independent and identically distributed random variables with common p.d.f.

$$f(x) = \begin{cases} \dfrac{1+\delta}{x^{2+\delta}}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \delta > 0.$$

Show that law of large numbers holds.

(a) Let $\{X_n, Y_n\}$, $n = 1, 2, \ldots$ be a sequence of random variables. Then

$$|X_n - Y_n| \xrightarrow{P} 0 \text{ and } Y_n \xrightarrow{\alpha} Y \Rightarrow X_n \xrightarrow{\alpha} Y.$$

Prove it.

(b) Let $\beta_n = E|X|^n < \infty$. Show that for arbitrary k, $2 \leq k \leq n$,

$$\beta_{k-1}^{\frac{1}{k-1}} \leq \beta_k^{\frac{1}{k}}$$

(c) Let (X, Y) be jointly distributed with p.d.f.

$$f(x, y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find marginal probability density functions of X and Y.

(d) Let $X_1, X_2, \ldots\ldots, X_n$ be i.i.d. random variables with common p.m.f.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \; k = 0, 1, 2, \ldots, n, \quad 0 < p < 1.$$

Obtain the p.m.f. of $S_m = X_1 + X_2 + \ldots + X_m$.

10×4=40

3. (a) Let X have a geometric distribution, then for any two non-negative integers m and n,

$$P(X > m + n / X > m) = P(X \geq n).$$

Prove it.

(b) Let X be a random variable with a continuous distribution function F. Show that $F(X)$ has the uniform distribution on $(0, 1)$.

(c) Let X be a Poisson random variable with parameter λ. Show that

$$P(X \le K) = \frac{1}{K!} \int_{\lambda}^{\infty} e^{-x} x^{K} \, dx.$$

(d) How large a sample must be taken in order that the probability will be at least 0·95 that $\overline{X}_n$ will be within 0·5 of μ. (μ is unknown and $\sigma = 1$). 10×4=40

4. (a) Let $F_n(x)$ be the distribution function defined by

$$F_n(x) = \begin{cases} 0, & \text{for } x \le -n \\ \dfrac{x+n}{2n}, & \text{for } -n < x < n \\ 1 & \text{for } x \ge n. \end{cases}$$

Is the $\lim_{n \to \infty} F_n(x)$ a distribution function ? If not, why ?

(b) If X_i can have only two values with equal probabilities i^{α} and $-i^{\alpha}$, show that law of large numbers can be applied to the independent variables $X_1, X_2, \ldots\ldots,$ if $\alpha < \frac{1}{2}$.

(c) If $A_1, A_2, \ldots\ldots,$ be a sequence of events on the probability space (S, B, P) and let $A = \overline{\lim}\, A_n$.

If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A) = 0$.

Prove it.

(d) Let $\{X_k\}$ be independent random variables with p.m.f.

$$P\left(X_k = k^{\lambda}\right) = P\left(X_k = -k^{\lambda}\right) = \frac{1}{2}, \quad k = 1, 2, \ldots .$$

Show that this sequence $\{X_k\}$ obeys central limit theorem, for $\lambda > 0$. $10\times4=40$

Section – B

5. Answer any *five* of the following : $8\times5=40$

(a) Show that for any discrete distribution, standard deviation is not less than mean deviation from mean.

(b) Let (X, Y) be jointly distributed with density function

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1; \quad 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Obtain correlation coefficient between X and Y.

(c) For the 2×2 table

a	*b*
c	*d*

prove that Chi-square test of independence gives

$$\chi^2 = \frac{N(ad - bc)}{(a+c)(b+d)(a+b)(c+d)},$$

$$N = a + b + c + d.$$

(d) The following data represent lifetimes (hours) of batteries for two different brands :

Brand A :	40	30	40	45	55	30
Brand B :	50	50	45	55	60	40

Test whether two brands are the same.

(e) Estimate U_2 from the following table :

x :	1	2	3	4	5
U_x :	7	–	13	21	37

(f) Evaluate $\log_e 7$ by Simpson's $\frac{1}{3}$rd rule.

6. (a) A cyclist pedals from his house to his college at a speed of 10 km per hour and back from the college to his house at 15 km per hour. Find the average speed.

(b) The random variables X and Y are jointly normally distributed and U and V are defined by $U = X\cos\alpha + Y\sin\alpha$, $V = Y\cos\alpha - X\sin\alpha$. Show that U and V will be uncorrelated if

$$\tan 2\alpha = \frac{2\gamma\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}.$$

(c) The following values of the function $f(x)$ for values of x are given :

$$f(1) = 4,\ f(2) = 5,\ f(7) = 5,\ f(8) = 4.$$

Find the value of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

(d) If third differences are constant, prove that

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{3}\left[f(0) + f\left(\frac{1}{\sqrt{2}}\right) + f\left(-\frac{1}{\sqrt{2}}\right)\right].$$

10×4=40

7. (a) The first three moments of a distribution about the value 1 are 2, 25 and 80. Find its mean, standard deviation and the moment-measure of skewness.

(b) You are working as a purchase manager for a company. The following information has been supplied to you by two manufacturers of electric bulbs.

	Company A	*Company B*
Mean life (in hours)	1300	1288
Standard deviation (in hours)	82	93
Sample size	100	100

Which brand of bulbs are you going to purchase if you desire to take a risk of 5% ?

(c) Describe clearly sign test. State its asymptotic relative efficiency with respect to t-test.

(d) The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable? 10×4=40

8. (a) Show that for discrete distribution $\beta_2 > 1$.

(b) Find the most likely price in Mumbai corresponding to the price of Rs. 70 at Kolkata from the following :

	Kolkata	*Mumbai*
Average price	65	67
Standard deviation	2·5	3·5

Correlation coefficient between the prices of commodities in the two cities is 0·8.

(c) Show that for t-distribution with n degrees of freedom, mean deviation about mean is given by

$$\frac{\sqrt{n}\,\Gamma\left[\left(\frac{n-1}{2}\right)\right]}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}.$$

(d) If $f(x)=\frac{1}{x^2}$, find the divided differences $f(a, b)$ and $f(a, b, c)$. 10×4=40